A Generalized Method for Calculating the RMS Values of Switching Power Converters

G. Th. Kostakis, Stefanos N. Manias, and Nikos I. Margaris

Abstract—In this paper an analytical method is presented which allows the exact calculation of the rms value of the state space variables of a switching power converter, whose starting point is the unified state-space representation of the switched networks and end result is a complete form of equations that include both the DC and RMS values. The method proposed in this paper, bridges the gap, which exists between the state-space technique and the small signal approximation that departures from the steady state ripple values are negligible, compared to the steady state values themselves. A new extended circuit model is proposed whose fixed topology contains all the elements of any DC-to-DC converter, regardless of its detailed configuration, and by which different converters can be characterized in the form of a table conveniently stored in a computer data bank to provide a useful tool for computer aided design and optimization. Finally, examples are given for Buck, Boost, and Buck–Boost regulator topologies.

Index Terms—Buck-boost regulator, capacitor, DC-to-DC converter, inductor, semiconductor.

NOMENCLATURE

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<td>C</td>
<td>Output capacitor.</td>
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<td>R</td>
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<td>T</td>
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<td>ton</td>
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<td>x(t)</td>
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<td>Xs</td>
<td>dc value of x(t).</td>
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<td>xrs(t)</td>
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<td>dc value of x(t).</td>
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<td>yd(t)</td>
<td>x(t) output variable.</td>
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<td>ed</td>
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<td>edr</td>
<td>rms value of ed(t).</td>
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<td>Vr</td>
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I. INTRODUCTION

SWITCHING power converters are inherently nonlinear and consequently it is very difficult to calculate the rms values of the state variable ripple using exact methods since the resulting differential equations cannot generally be solved. Instead, small-signal methods are commonly used, where one linearizes the power converter model about a quiescent operating point [1], [2]. Unfortunately, because of the small-signal approximation, these methods provide a sense of absence of the switching converter variables rms values [3] and as an example the rms value of a capacitor current is considered to be zero due to the fact that the dc ripple of the capacitor voltage is zero. So despite the fact that there is zero ripple voltage across the capacitor, there is rms current that flows through the capacitor.

In this proposed work the authors remain strictly in the domain of the state-space equations of the two-switched models providing the design engineer with physical insight into the behavior of the original switched circuit and a significant tool to calculate the rms values of the power converter variables. These rms values are important in order to calculate the current stresses of the different power converter devices as well as to filter design in order to meet the given specifications.

II. PROPOSED METHOD

Any switching dc-to-dc converter operating in the continuous conduction mode can be described by the state space equations of the two-switched model [1]

\[ \dot{x}(t) = A_1 x(t) + B_1 V_i \]
\[ y(t) = C^T_1 x(t) \] during the ON interval
\[ \dot{x}(t) = A_2 x(t) + B_2 V_i \]
\[ y(t) = C^T_2 x(t) \] during the OFF interval

where for second order system

\[ x(t) = (x_1(t), x_2(t))^T \]
\[ y(t) = (y_1(t), y_2(t))^T \]
\[ A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \]
\[ B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \]
\[ C^T_1 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad C^T_2 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \]

\[ \det(A_1) \neq 0, \quad \det(A_2) \neq 0, \quad \text{trace}(A_1) < 0, \quad \text{trace}(A_2) < 0 \]
Moreover, the singular points of (1) and (2) are given, respectively, by

\[ \hat{x}_1 = \left( \begin{array}{c} \hat{x}_{11} \\ \hat{x}_{12} \end{array} \right) = -A_1^{-1}B_1V_i \]  
(6)

\[ \hat{x}_2 = \left( \begin{array}{c} \hat{x}_{21} \\ \hat{x}_{22} \end{array} \right) = -A_2^{-1}B_2V_i \]  
(7)

The coefficient matrices for the three well-known topologies (i.e., Boost, Buck–Boost and Buck) are given in the Appendix which must satisfy the following relations.

1) Boost converter topology, \( B_1 = B_2 \).
2) Buck–Boost converter topology, \( B_2 = 0 \).
3) Buck converter topology, \( A_1 = A_2, B_2 = 0, C_1^T = C_2^T \).

If we assume that the duty ratio \( d \) is constant from cycle to cycle, namely, \( d = D \) (steady state dc duty ratio), and \( D' = 1 - D \), then the steady-state (dc) model is obtained as follows:

The solution of the differential (1) at the end of the turn-on time is given by

\[ x(DT) = e^{TA_1D}x(0) + (I - e^{TA_1D})\hat{x}_1. \]  
(8)

Also, the solution of the differential (2) at the end of the turn-off time is given by

\[ x(T) = e^{TA_2D'}x(DT) + (I - e^{TA_2D'})\hat{x}_2. \]  
(9)

Substituting (8) into (9) then the initial conditions of the differential equation (1) and (2) can be found as follows

**Quiescent operating point \( P \) at turn-on**

\[ \begin{align*}
P &= \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^T \\
&= x(0) \\
&= x(T) \\
&= (I - e^{TA_2D'}e^{TA_1D})^{-1}(I - e^{TA_2D'})^{\ast}(x_2 - \hat{x}_1) + \hat{x}_1.
\end{align*} \]  
(10)

Also, substituting (9) into (8) yields to

**Quiescent operating point \( Q \) at turn-off**

\[ \begin{align*}
Q &= \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}^T \\
&= x(DT) \\
&= (I - e^{TA_2D'}e^{TA_1D})^{-1}(I - e^{TA_2D'})^{\ast}(x_2 - \hat{x}_1) + \hat{x}_2.
\end{align*} \]  
(11)

Fig. 1 shows an example of convergence of a Buck power switching converter at its respective \( P \) and \( Q \) operating points under constant duty cycle and Fig. 2 shows the trajectories of quiescent operating points \( P \) and \( Q \) in the state space for Boost converter. Since the state vector \( x(t) \) is periodic its dc value is found from:

\[ X = \frac{1}{T} \int_0^T x(t) dt \]  
(12)

Therefore, substituting (1), (2), (6), (7), into (12) the following expression is found:

\[ \begin{align*}
X &= \frac{1}{T} \int_0^{t_{cm}} x(t) dt + \frac{1}{T} \int_{t_{cm}}^T x(t) dt \\
&= \frac{1}{T} \int_0^{t_{cm}} (A_1^{-1}x_1 + \hat{x}_1) dt + \frac{1}{T} \int_{t_{cm}}^T (A_2^{-1}x + \hat{x}_2) dt \\
&= \frac{A_1^{-1}}{T} \int_0^{t_{cm}} \dot{x}_1 dt + \frac{t_{cm}}{T} + \frac{A_2^{-1}}{T} \int_{t_{cm}}^T \dot{x}_2 dt \\
&= (A_1^{-1} - A_2^{-1}) \frac{Q - P}{T} + D\hat{x}_1 + D'\hat{x}_2
\end{align*} \]  
(13)

and consequently

\[ \begin{align*}
Y &= (C_1^T A_1^{-1} - C_2^T A_2^{-1}) \left( \frac{Q - P}{T} \right) + DC_1^T \hat{x}_1 + D'C_2^T \hat{x}_2
\end{align*} \]  
(14)

By definition the rms value of a periodic function \( x(t) \) is given by

\[ \hat{X} = \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt} \]

or

\[ \hat{X}^2 = \frac{1}{T} \int_0^T [x(t)]^2 dt \]  
(15)

Our objective now is to replace the conventional state-space equations by a new set of equations from which one can be able to calculate both the dc and rms values of the state variables of a switching topology. In order to obtain these equations a
new extended state vector $w(t)$ and an output vector $u(t)$ are introduced as follows:

$$w(t) = (x_1(t) \ x_2(t) \ (x_1(t))^2 \ (x_2(t))^2 \ x_1(t) \ x_2(t))^T \ (16)$$

$$u(t) = (y_1(t) \ y_2(t) \ (y_1(t))^2 \ (y_2(t))^2)^T \ (17)$$

Examining (16) and (17) becomes obvious that the calculation of the dc values of $w(t)$ and $u(t)$ will contain both the dc and rms values of the variables, which can be expressed theoretically by the following equations:

$$W = (X_1 \ X_2 \ X_1^2 \ X_2^2 \ x_1 \ x_2)^T \ (18)$$

$$V = (Y_1 \ Y_2 \ Y_1^2 \ Y_2^2)^T \ (19)$$

where $x_{1} \ x_{2}$ denotes the dc value of the product $x_1(t) \ x_2(t)$.

Next, taking the derivative of (16) for the ON interval and using (1) and (3) yields to

$$\dot{w}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ (x_1(t))^2 \\ (x_2(t))^2 \\ x_1(t) \ x_2(t) \\ x_1(t) \ x_2(t) + x_2(t) \ x_1(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 2b_1V_i & 0 & a_{11} & 0 & a_{12} \\ 0 & 2b_2V_i & 0 & 2a_{22} & 2a_{21} \\ b_2V_i & b_1V_i & a_{21} & a_{12} & (a_{11} + a_{22}) \end{pmatrix} \ x_1(t) \ x_2(t) + x_2(t) \ x_1(t)$$

$\dot{u}(t) = A_{1w} \ w(t) + B_{1w}$ during the ON interval \hspace{1cm} (20)

Respectively, using (17) and (4) the output vector $u(t)$ is given by

$$u(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ (y_1(t))^2 \\ (y_2(t))^2 \end{pmatrix} = \begin{pmatrix} c_{11}x_1(t) + c_{12}x_2(t) \\ c_{21}x_1(t) + c_{22}x_2(t) \\ (c_{11}x_1(t) + c_{12}x_2(t))^2 \\ (c_{21}x_1(t) + c_{22}x_2(t))^2 \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{11}^2 & c_{12}^2 & 2c_{11}c_{12} \\ 0 & 0 & c_{21}^2 & c_{22}^2 & 2c_{21}c_{22} \end{pmatrix} \ \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_1(t) \ x_2(t) + x_2(t) \ x_1(t) \end{pmatrix}$$

$\dot{u}(t) = A_{2w} \ w(t) + B_{2w}$ during the ON interval \hspace{1cm} (21)

Finally, using the same procedure the new set of state space equations are obtained for the OFF interval which can describe any switching dc-dc converter operating in the continuous conduction mode as follows:

$$\dot{u}(t) = A_{3w} \ w(t) + B_{3w}$$

$u(t) = F_{3w}^T \ w(t)$ during the OFF interval \hspace{1cm} (22)
where the coefficient matrices are extension of the existing matrices and are given by

\[
A_{2_{w}} = \begin{pmatrix}
g_{11} & g_{12} & 0 & 0 & 0 \\
g_{21} & g_{22} & 0 & 0 & 0 \\
2h_{11}V_{i} & 0 & 2g_{11} & 0 & 2g_{12} \\
0 & 2h_{22}V_{i} & 0 & 2g_{22} & 2g_{21} \\
h_{12}V_{i} & h_{11}V_{i} & g_{21} & g_{12} & g_{11}+g_{22}
\end{pmatrix}
\]

and the dc values for the new extended vectors \(w(t)\) and \(v(t)\) are found from (13) and (14) to be

\[
W = (W_{1} W_{2} W_{3} W_{4} W_{5})^{T}
\]

\[
=W\left(\begin{pmatrix}
A_{2_{w}}^{-1} - A_{2_{w}}^{-1} \left(\begin{array}{cc}
Q_{w} & -P \end{array}\right) T
\end{pmatrix}
\end{pmatrix}
\]

\[
V = (V_{1} V_{2} V_{3} V_{4})^{T}
\]

Moreover, the new singular points \(\hat{w}_{1}\) and \(\hat{w}_{2}\) are defined and given by

\[
\hat{w}_{1} = \begin{pmatrix}
x_{1}(0) \\
x_{2}(0)
\end{pmatrix} = \begin{pmatrix}
P_{1} \\
P_{2}
\end{pmatrix}
\]

\[
\hat{w}_{2} = \begin{pmatrix}
x_{2}(0) \\
x_{2}(0)
\end{pmatrix} = \begin{pmatrix}
P_{2} \\
P_{2}
\end{pmatrix}
\]

Therefore, using (26) and (27) a new set of equations are obtained from which one can calculate the rms values of any switching system variables and their respective rms ripple. These set of equations are given as follows:

\[
\begin{align*}
\sqrt{\text{Rms values}} &= \begin{pmatrix}
\hat{X}_{1}^{2} = W_{3} \\
\hat{X}_{2}^{2} = W_{4} \\
\hat{Y}_{1}^{2} = V_{3} \\
\hat{Y}_{2}^{2} = V_{4}
\end{pmatrix} \\
\text{Ripple rms values} &= \begin{pmatrix}
\hat{x}_{1}^{2} = \hat{X}_{1}^{2} - X_{1}^{2} = W_{3} - W_{1} \\
\hat{x}_{2}^{2} = \hat{X}_{2}^{2} - X_{2}^{2} = W_{4} - W_{2} \\
\hat{y}_{1}^{2} = \hat{Y}_{1}^{2} - Y_{1}^{2} = V_{3} - V_{1} \\
\hat{y}_{2}^{2} = \hat{Y}_{2}^{2} - Y_{2}^{2} = V_{4} - V_{2}
\end{pmatrix}
\end{align*}
\]

III. AVERAGING THE PROPOSED METHOD

Applying the state space averaging method to the proposed state space model [see (20)–(22)] then the dc state vector \(W_{AV}\) and the dc output vector \(V_{AV}\) become independent of \(T\) and storage elements. Knowing that for the averaging method

\[
\lim_{T \to 0} \frac{Q-P}{T} = DA_{1}(X_{AV} - \hat{x}) = -DF_{1}\hat{x}_{1} - DF_{2}\hat{x}_{2}
\]

then the respective averaging results of the proposed method are conveniently found by (26) and (27) and are expressed as

\[
\begin{align*}
W_{AV} &= -[DA_{2_{w}} + DF_{1}\hat{x}_{1} + DF_{2}\hat{x}_{2}] \\
V_{AV} &= (DF_{1} + DF_{2}^{T})W
\end{align*}
\]

Finally, Table I provides the results which are found in this section.

Table I contains the equations that can be used in a CAD environment in order to exactly calculate the rms values of a switching converter and also contains the respective averaging equations if someone wishes to obtain results by direct hand calculation. It is obvious that when the averaging method is used is not an exact calculation since the rms values become independent of the \(L\) and \(C\) elements of the power converter.

A. Example: Boost Power Stage with Parasitics

In this section we will now illustrate the application of the proposed method to the boost power converter which is shown in Fig. 3.

If one defines

\[
\begin{align*}
\hat{w}_{1} &= -A_{2_{w}}^{-1}B_{w_{1}} \\
\hat{w}_{2} &= -A_{2_{w}}^{-1}B_{w_{2}}
\end{align*}
\]

\[
= \begin{pmatrix}
x_{11} & x_{12} & x_{11} & x_{12} & x_{11} \cdot x_{12}
\end{pmatrix}^{T}
\]

\[
= \begin{pmatrix}
x_{21} & x_{22} & x_{21} & x_{22} & x_{21} \cdot x_{22}
\end{pmatrix}^{T}
\]

From (5) one can conclude that \(\det(A_{2_{w}}) = 4 \cdot \text{trace}(A_{2_{w}}) \cdot (\det(A_{2_{w}}))^{2} \neq 0\) and the new singular points \(\hat{w}_{1}\) and \(\hat{w}_{2}\) are defined and given by

\[
\hat{w}_{1} = -A_{2_{w}}^{-1}B_{w_{1}}
\]

\[
= \begin{pmatrix}
i_{L}(t) \\
v_{C}(t)
\end{pmatrix} \quad \hat{v}(t) = \begin{pmatrix}
v_{o}(t) \\
(i_{L}(t))^{2} \\
(v_{C}(t))^{2} \\
i_{L}(t) \cdot v_{C}(t)
\end{pmatrix}
\]

(41)
then using the (39) and (40) the dc values of the vectors are expressed as follows

\[
W_{AV} = \begin{pmatrix}
\frac{R + r_p}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
\frac{(R + r_p)RD'}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
\frac{R}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
\frac{R^2D}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
\frac{(R + r_p)RD'}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
\end{pmatrix}
\]

where the coefficient matrices are given in the Appendix.

Therefore, using (42) and (43) we obtain the following novel equations for the boost converter:

\[
V_{AV} = \begin{pmatrix}
I_L \\
V_C \\
\bar{I}_L \\
V_C \\
I_L \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(R + r_p)RD' \\
r_s(R + r_p) + Rr_pD + R^2D^2 V_i \\
R^2D \\
(r_s(R + r_p) + Rr_pD + R^2D^2 V_i) \cdot DD' \\
\end{pmatrix}
\]

TABLE I

**SYNOPSIS OF THE PROPOSED METHOD**

<table>
<thead>
<tr>
<th>Proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P = (I - e^{\lambda D})^2(I - e^{\lambda D}) \cdot x + \hat{x} )</td>
</tr>
<tr>
<td>( Q = (I - e^{\lambda D}) \cdot y + \hat{y} )</td>
</tr>
<tr>
<td>( X = (A_1^{-1} - A_2^{-1}) \cdot \frac{Q - P}{T} + D \hat{x} )</td>
</tr>
<tr>
<td>( Y = (C_1^{-1} - C_2^{-1}) \cdot \frac{Q - P}{T} + D \hat{y} )</td>
</tr>
<tr>
<td>( W = (A_w^{-1} - A_w^{-1}) \cdot \frac{Q - P}{T} + D \hat{w} )</td>
</tr>
</tbody>
</table>

**Averaging the Proposed Method**

\[
P = Q - X_{AV} \\
X_{AV} = \left[ DA_x + D'A_{xw} \right] \cdot \left[ DB_x + D'B_{xw} \right] \\
V_{AV} = \left[ DC_x' + D'C_{xw}' \right] \cdot X_{AV} \\
W_{AV} = \left[ D F_x' + D'F_{xw}' \right] \cdot W_{AV}
\]

![Fig. 3. Boost power stage with parasitics included.](image)

**Fig. 3.** Boost power stage with parasitics included.

Fig. 4 shows the results of (46) from which one can obtain useful information for the output capacitor rms current as a function of the switch duty cycle and the load resistance.

Moreover, the following equations are obtained which are in agreement with reference [1]

\[
I_L = \frac{R + r_p}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
V_C = \frac{(R + r_p)RD'}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i} \\
V_o = V_C = \frac{1}{D} \frac{R^2D^2}{r_s(R + r_p) + Rr_pD + R^2D^2 V_i}
\]
Using (45) and (46) the following equation is found

\[ \dot{\bar{I}}_o = r_P^2 \bar{C} \] (49)

Finally, using (44)–(49) one can verify the specifications of the boost converter, calculate the ratings of the input inductor and the output capacitor and assist to obtain the ratings of the switching devices.

B. Derivation of the Boost Power Stage Equations

The \( V_o \) to \( V_i \) dc transfer ratio \( H \) is obtained from (48) as follows:

\[ H(D) = \frac{V_o}{V_i} = \frac{(R+r_p)R(1-D)}{r_s(R+r_p)+Rr_p(1-D)+R^2(1-D)^2} \] (50)

Differentiating (50) with respect to \( D \) and taking the first derivative equal to zero results in to:

\[ D_{1,2} = 1 \pm \sqrt{\frac{r_s}{R} \left( 1 + \frac{r_p}{R} \right)} \] (51)

The root \( D_1 = 1 + \sqrt{(r_s/R)(1+(r_p/R))} \) is rejected because is greater than 1 and the second root \( D_2 \) is accepted if is positive i.e.:

\[ 1 - \sqrt{\frac{r_s}{R} \left( 1 + \frac{r_p}{R} \right)} > 0 \]

which implies

\[ R > \frac{r_s}{2} + \sqrt{\left( \frac{r_s}{2} \right)^2 + r_s r_p} \] (52)

A corresponding expression for the output power, \( P_o \), can now be written as follows:

\[ \frac{\dot{V}_o^2}{P_o} > \frac{r_s}{2} + \sqrt{\left( \frac{r_s}{2} \right)^2 + r_s r_p} \]

or

\[ P_o < \frac{\dot{V}_o^2}{\frac{r_s}{2} + \sqrt{\left( \frac{r_s}{2} \right)^2 + r_s r_p}} \] (53)

In order for the boost action to occur the inequality (52) must be satisfied.

Moreover, the maximum value of \( H \) results when \( D = D_2 \) and is found from (50) to be:

\[ H_{\max} = H(D_2) = \frac{R}{2 \sqrt{\frac{r_s}{R} \left( 1 + \frac{r_p}{R} \right)}} + \frac{r_p}{R} \] (54)

The efficiency \( \eta \) of the Boost converter can be determined as follows

\[ \eta = \frac{P_o}{P_{in}} = \frac{\dot{V}_o^2}{V_i I_L} = \frac{1}{R \dot{V}_i} \frac{V_3}{W_1} \] (55)

Substituting (44) and (47) into (55) then the efficiency becomes independent of \( T \), namely

\[ \eta = \frac{1}{R \dot{V}_i} \frac{\frac{R^2((R+r_p)^2 - R(R+2r_p)D')}{(r_s(R+r_p)+Rr_pD'+R^2D'^2)^2} \dot{V}_i^2}{v'} \]

\[ = \frac{R}{R + r_p} \frac{((R+r_p)^2 - R(R+2r_p)D')}{r_s(R+r_p)+Rr_pD'+R^2D'^2} \] (56)

This expression is plotted in Fig. 5, again for several values of \( R \).
In case of a Buck or Buck–boost converter where the input current is equal to the inductor current for the ON interval, the efficiency $\eta$ is given by

$$\eta = \frac{P_o}{P_{in}} = \frac{\bar{V}_3}{X_1 V_i} = \frac{\bar{V}_3}{V_i} = \frac{1}{R V_i} \left[ \int_{0}^{t_{on}} (A^{-1}_1 \dot{x} + \dot{x}_1) dt \right]$$

where $X_1$ = inductor current dc value and $\bar{V}_3$ = output voltage rms value.

If averaging is applied to (57) then the efficiency becomes independent of $T$, namely

$$\eta = \frac{1}{R V_i} \left[ A_1^{-1} \left( D A_1 (X_{AV} - \dot{x}_1) + D \dot{x}_1 \right) \right]$$

Finally, Table II provides also the relations obtained in this section for Buck–Boost and Buck topologies.

**TABLE II**

<table>
<thead>
<tr>
<th>AVERAGING RESULTS FOR BUCK–BOOST AND BOOST TOPOLOGIES</th>
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<table>
<thead>
<tr>
<th>Buck-Boost</th>
<th>Boost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_o = \frac{R + r_s \sqrt{2} - R (R + 2 r_s) V'}{r_s (R + r_s) + r_s D'} V_s$</td>
<td>$V_o = \frac{R + r_s \sqrt{2} - R (R + 2 r_s) V'}{r_s (R + r_s) + r_s D'} V_s$</td>
</tr>
<tr>
<td>$\bar{V}_o = \frac{r_s D'}{r_s (R + r_s) + r_s D'} \bar{V}_s$</td>
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</tr>
<tr>
<td>$\bar{V}_x = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_y$</td>
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</tr>
<tr>
<td>$\bar{V}_y = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_y$</td>
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</tr>
<tr>
<td>$\bar{V}_z = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_z$</td>
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</tr>
<tr>
<td>$\bar{V}_s = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_s$</td>
<td>$\bar{V}_s = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_s$</td>
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<tr>
<td>$\bar{V}_y = \frac{R + r_s}{r_s (R + r_s) + r_s D'} \bar{V}_y$</td>
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</tr>
<tr>
<td>$\eta = \frac{R + r_s \sqrt{2} - R (R + 2 r_s) V'}{r_s (R + r_s) + r_s D'} \eta$</td>
<td>$\eta = \frac{R + r_s \sqrt{2} - R (R + 2 r_s) V'}{r_s (R + r_s) + r_s D'} \eta$</td>
</tr>
</tbody>
</table>

**IV. CONCLUSIONS**

In this paper a general method for the exact calculation of the rms values of any switching DC-to-DC converter has been developed through an extended state-space model. The fundamental step of the proposed method was to extend the standard state-space equations of the two-switched networks so that both the DC and RMS values of the state space variables to be included. For a given frequency, one can find the operating points $P$ and $Q$ (see Table I.) and, using (33)–(40) plots the time de-
pendence of the newly created variables during a period $T$ to obtain the steady state switching ripple. Although the state-space modeling has been developed in this paper for second order switching converters, the method can be extended to $n$th order system. In this work examples are given for Boost, Buck–Boost and Buck power topologies. Finally, using this method the design engineer having the specifications and the components of the power converter topology can calculate exactly their rms values, which can assist to perform optimization of the power converter components.

In the Appendix it is demonstrated analytically for the three power stages the state space averaged results.

**APPENDIX**

A. Boost State Space Equations (see Fig. 3.)

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} &= \begin{pmatrix} -\frac{r_p}{L} & 0 \\ 0 & -\frac{1}{(R+r_p)C} \end{pmatrix} \cdot \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_i \\
&\text{during the ON interval}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \dot{v}_L(t) \\ \dot{i}_C(t) \end{pmatrix} &= \begin{pmatrix} \frac{R}{R+r_p} & -\frac{1}{(R+r_p)C} \\ 0 & \frac{1}{R} \end{pmatrix} \cdot \begin{pmatrix} \dot{v}_L(t) \\ \dot{i}_C(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_i \\
&\text{during the OFF interval}
\end{align*}
\]

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} V_i \\ r_s \end{pmatrix}^T \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} V_i \\ r_s \end{pmatrix}^{R+r_s} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 0
\]

B. Buck Boost State Space Equations (see Figs. 6 and 7)

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} &= \begin{pmatrix} -\frac{r_p}{L} & 0 \\ 0 & -\frac{1}{(R+r_p)C} \end{pmatrix} \cdot \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_i \\
&\text{during the ON interval}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \dot{v}_L(t) \\ \dot{i}_C(t) \end{pmatrix} &= \begin{pmatrix} \frac{R}{R+r_p} & -\frac{1}{(R+r_p)C} \\ 0 & \frac{1}{R} \end{pmatrix} \cdot \begin{pmatrix} \dot{v}_L(t) \\ \dot{i}_C(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_i \\
&\text{during the OFF interval}
\end{align*}
\]

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} V_i \\ r_s \end{pmatrix}^T \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} V_i \\ r_s \end{pmatrix}^{R+r_s} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 0
\]
Therefore, using (59) and (60) we obtain the following novel equations of the Buck–Boost converter:

\[
\begin{align*}
\tilde{V}_o &= \text{rms value of the output voltage} \\
&= \frac{R + r_p}{r_s(R + r_p) + R r_p D' + R^2 D'^2} \frac{D V_i}{D' V_i} \\
\tilde{v}_o &= \text{rms value of the output voltage ripple} \\
&= \frac{R r_p \sqrt{D' D^2}}{r_s(R + r_p) + R r_p D' + R^2 D'^2} D V_i \\
\tilde{i}_C &= \text{rms value of the output capacitor current ripple} \\
&= \frac{R D \sqrt{D' D^2}}{r_s(R + r_p) + R r_p D' + R^2 D'^2} V_i \\
\end{align*}
\]

and the well known equations

\[
\begin{align*}
I_L &= \frac{R + r_p}{r_s(R + r_p) + R r_p D' + R^2 D'^2} D V_i \\
V_o &= V_C = \frac{(R + r_p) R}{r_s(R + r_p) + R r_p D' + R^2 D'^2} D' D V_i \\
\end{align*}
\]

The efficiency \( \eta \) of the Buck Boost converter is given by (58)

\[
\eta = \frac{R}{R + r_p} \frac{(R + r_p)^2 - R(R + 2 r_p) D}{R + r_p} D' 
\]

C. Buck State Space Equations (see Fig. 7.)

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} &= \begin{pmatrix} \frac{R}{R + r_p} R & \frac{R}{r_s(R + r_p) + R r_p D' + R^2 D'^2} \\ 0 & \frac{R}{r_s(R + r_p) + R r_p D' + R^2 D'^2} \end{pmatrix} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{R}{R + r_p} \frac{R}{(R + r_p) L} & \frac{R}{(R + r_p) C} \\ \frac{R}{(R + r_p) C} & \frac{1}{(R + r_p) C} \end{pmatrix} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{R}{R + r_p} \frac{R}{R + r_p} \\ \frac{1}{R + r_p} \end{pmatrix} \begin{pmatrix} i_L(t) \\ v_C(t) \end{pmatrix}
\end{align*}
\]

during the ON interval
Results:

\[ \frac{d}{dt} \left( \frac{i_L(t)}{v_C(t)} \right) = \left( \begin{array}{c} -\frac{R}{L} \frac{r_p}{R+r_p} \frac{V_i}{L(R+r_p)} \\ \frac{R}{(R+r_p)C} \frac{1}{(R+r_p)C} \end{array} \right) \]

\[ I_L = \frac{D}{R+r_s} V_i \]

\[ V_o = V_C = \frac{RD}{R+r_s} V_i \]

\[ \eta = \frac{R}{R+r_s} \]

\[ * \mathbf{x}_1 = \left( \begin{array}{c} V_i \\ \frac{V_i R}{R+r_s} \end{array} \right) T \quad * \mathbf{x}_2 = \mathbf{0} \]

\[ W_{AV} = \left( \begin{array}{cccc} \frac{D}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ \frac{D}{R+r_s} V_i^2 & \frac{RD}{R+r_s} V_i^2 \end{array} \right) \]

\[ \mathbf{V}_{AV} = \left( \begin{array}{cccc} I_L & V_C & I_L^2 & V_C^2 \\ \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} \end{array} \right) \]

\[ \mathbf{V}_o = \left( \begin{array}{cccc} 0 & 0 & \frac{RD}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ 0 & 0 & 0 & 0 \end{array} \right) \]

\[ \dot{\mathbf{V}}_o = \dot{V}_o \dot{\mathbf{v}}_o = \mathbf{0} \quad \dot{\mathbf{i}}_C = \mathbf{0} \quad \dot{\mathbf{i}}_L = \mathbf{0} \quad \dot{\mathbf{v}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \quad \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ * \mathbf{x}_1 = \left( \begin{array}{c} V_i \\ \frac{V_i R}{R+r_s} \end{array} \right) T \quad * \mathbf{x}_2 = \mathbf{0} \]

\[ W_{AV} = \left( \begin{array}{cccc} \frac{D}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ \frac{D}{R+r_s} V_i^2 & \frac{RD}{R+r_s} V_i^2 \end{array} \right) \]

\[ \mathbf{V}_{AV} = \left( \begin{array}{cccc} I_L & V_C & I_L^2 & V_C^2 \\ \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} \end{array} \right) \]

\[ \mathbf{V}_o = \left( \begin{array}{cccc} 0 & 0 & \frac{RD}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ 0 & 0 & 0 & 0 \end{array} \right) \]

\[ \dot{\mathbf{V}}_o = \dot{V}_o \dot{\mathbf{v}}_o = \mathbf{0} \quad \dot{\mathbf{i}}_C = \mathbf{0} \quad \dot{\mathbf{i}}_L = \mathbf{0} \quad \dot{\mathbf{v}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ * \mathbf{x}_1 = \left( \begin{array}{c} V_i \\ \frac{V_i R}{R+r_s} \end{array} \right) T \quad * \mathbf{x}_2 = \mathbf{0} \]

\[ W_{AV} = \left( \begin{array}{cccc} \frac{D}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ \frac{D}{R+r_s} V_i^2 & \frac{RD}{R+r_s} V_i^2 \end{array} \right) \]

\[ \mathbf{V}_{AV} = \left( \begin{array}{cccc} I_L & V_C & I_L^2 & V_C^2 \\ \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} & \frac{I_L}{V_C} \end{array} \right) \]

\[ \mathbf{V}_o = \left( \begin{array}{cccc} 0 & 0 & \frac{RD}{R+r_s} V_i & \frac{RD}{R+r_s} V_i \\ 0 & 0 & 0 & 0 \end{array} \right) \]

\[ \dot{\mathbf{V}}_o = \dot{V}_o \dot{\mathbf{v}}_o = \mathbf{0} \quad \dot{\mathbf{i}}_C = \mathbf{0} \quad \dot{\mathbf{i}}_L = \mathbf{0} \quad \dot{\mathbf{v}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ \dot{\mathbf{I}}_C = \mathbf{0} \quad \dot{\mathbf{I}}_L = \mathbf{0} \]

\[ \dot{\mathbf{v}}_C = \mathbf{0} \]

\[ * \mathbf{x}_1 = \left( \begin{array}{c} V_i \\ \frac{V_i R}{R+r_s} \end{array} \right) T \quad * \mathbf{x}_2 = \mathbf{0} \]